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(Restricted) Quantized Enveloping Algebras of Simple Lie superalgebras and Universal R-Matrices

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In this note, we difine a (Jimbo type) quantized enveloping superalgebras $U_q(G)$ of complex simple Lie superalgebras **G** of types A, B, C, D (all types) and types F_4 and G_3 (distinguished types). We can get a defininig relations of $U_q(G)$, which are consist of q-Serre relations and *additional relations* . They were unknown even if $q=1$. Moreover we define a restricted quantum groups $u_\zeta(G)$ at a root of unity ζ . Finally, we consider a *Hopf algebricazation* $u_\zeta(G)^\sigma$ of the Hopf superalgebra $u_\zeta(G)$, and construct the universal R-matrix of $u_\zeta(G)^\sigma$. Our construction is due to Drinfeld's quantum double construction. By using quantum double construction, we can also show a Poincaré-Birkhoff-Witt type theorem for $U_q(G)$ and $u_\zeta(G)$.

In [Y1-2], we introduced the (Drinfeld type) quantized enveloping superalgebras $U_h(G)$, showed that $U_h(G)$ is an h-adic topologically free **C**[[h]]-Hopf algebra, and gave an explicit formula of universal R-matrix of $U_h(G)^\sigma$. The arguments used in this note are the essentially same arguments as we used in [Y2].

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§1.Quantum double construction.

Let **K** be a field. Suppose char (**K**) = 0.Let (A,Δ,S,ε) is **K**-Hopf algebras with coproduct $\Delta : A \to A \otimes A$, antipode, $S : A \to A$ and counit $\varepsilon : A \to$ **K** . Moreover we assume that there is a symmetric Hopf-pairing $< , > : A \otimes A \to$ **K** , namely $< , >$ is a symmetric **K**-bilinear form such that

(1) $<\Delta(x), y \otimes z> = <x, yz>$,
(2) $<S(x), y> = <x, S(y)>$,
(3) $<1, x> = \varepsilon(x)$

where $x, y, z \in A$.

We call a Hopf-algebra $A^{op} = (A, \Delta^{op}, S, \varepsilon)$ the *opposite* Hopf-algebra of A where $\Delta^{op} = \tau \circ \Delta$ and $\tau(x \otimes y) = y \otimes x$.

Proposition 1.1. (*Quantum double*) There is a unique **K**-Hopf algebra $(D = D(A), \Delta_D, S_D, \varepsilon_D)$ satisfying:

(1) As **K**-vector spaces, $D \cong A \otimes A$.

(2) The **K**-linear maps $A \to A \otimes A$ $(x \to x \otimes 1)$ and $A^{op} \to A \otimes A$ $(x \to 1 \otimes x)$ are homomorphisms of Hopf-algebras.

(3) The product of D is defined as follows; if $x, y \in A$ and $\Delta^{(2)}(x) = \Sigma_i x_i^{(1)} \otimes x_i^{(2)} \otimes x_i^{(3)}$ and $\Delta^{(2)}(y) = \Sigma_i y_i^{(1)} \otimes y_i^{(2)} \otimes y_i^{(3)}$, then

$$(v \otimes x) \cdot (y \otimes w) = \Sigma_{i,j} <x_i^{(1)}, y_j^{(3)}><x_i^{(3)}, S(y_j^{(1)})>(v\, y_j^{(2)} \otimes x_i^{(2)} w).$$

Proposition 1.2. (*Universal R-matrix of D* (A)) Assume that $\dim A < \infty$ and $<,>$ is non-degenerate. Let $\{e_i\}$ and $\{e^i\}$ are two bases of A such that $<e_i, e^j> = \delta_{ij}$. Then $\boldsymbol{R} = \Sigma_i (e_i \otimes 1) \otimes (1 \otimes e^i) \in D \otimes D$ satisfies:

(0) $\boldsymbol{R}^{-1} = (1 \otimes S^{-1})(\boldsymbol{R})$.

(1) $\boldsymbol{R}\, \Delta_D(a) \boldsymbol{R}^{-1} = \Delta_D{}^{op}(a)$ $(a \in D)$.

(2) $(1 \otimes \Delta_D)(\boldsymbol{R}) = \boldsymbol{R}_{13}\boldsymbol{R}_{12}$, $(\Delta_D \otimes 1)(\boldsymbol{R}) = \boldsymbol{R}_{23}\boldsymbol{R}_{13}$.

Remark. From (1) and (2), we can easily see that $\boldsymbol{R}$ satisfies the *Yang-Baxter equation* :

$$\boldsymbol{R}_{12}\boldsymbol{R}_{13}\boldsymbol{R}_{23} = \boldsymbol{R}_{23}\boldsymbol{R}_{13}\boldsymbol{R}_{12}.$$

Therefore $\boldsymbol{R}$ is called the *universal R-matrix of* D.

§2. Quantized enveloping (super)algebras.

Here we give an abstruct definition of *Quantized enveloping (super)algebras* by using the *Quantum double construction.*

Let $\mathbb{E}$ be an N-dimensional **K**-vector space. Assume that there is a non-degenerate bi-linear form $(\ ,\ ) : \mathbb{E} \times \mathbb{E} \to \mathbf{K}$ with a basis $\{\underline{\varepsilon}_i \mid 1 \le i \le N\}$ such that $(\underline{\varepsilon}_i, \underline{\varepsilon}_j) = 0\ (i \ne j)$, $(\underline{\varepsilon}_i, \underline{\varepsilon}_i) \in \mathrm{Z} - \{0\}$. Let $\Pi = \{\alpha_i \in \mathbb{E} \mid 1 \le i \le n\}$ be the set of linearly independent elements. Suppose that $(\alpha_i, \alpha_j) \in (1/4)\mathrm{Z}$. Let $p : \Pi \to \mathbf{Z}/2\mathbf{Z} = \{0, 1\}$ be the function. Write $p(i)$ for $p(\alpha_i)$. We call p the *parity function*. Put $P_+ = \mathbf{Z}\underline{\varepsilon}_1 \oplus \cdots \oplus \mathbf{Z}\underline{\varepsilon}_N$.

Let $q \in \mathbf{K}^\times$. Let $U^\sim_q b_+^\sigma$ be a **K**-algebra with generators $\{E_i\ (1 \le i \le n), K_\lambda\ (\lambda \in P_+)\ ,\sigma\}$ and defining relations:

$(U^\sim.1)$ $\sigma^2 = 1$, $\sigma E_i \sigma = (-1)^{p(i)} E_i$, $\sigma K_\lambda \sigma = K_\lambda$,

$(U^\sim.2)$ $K_0 = 1$, $K_\lambda K_\mu = K_{\lambda+\mu}$ $(\lambda, \mu \in P_+)$,

$(U^\sim.3)$ $K_\lambda E_i K_\lambda^{-1} = q^{(\alpha_i,\lambda)} E_i$.

Moreover $U^\sim_q b_+^\sigma$ has a **K**-Hopf algebra such that

$(U^\sim.4)$ $\Delta(\sigma) = \sigma \otimes \sigma$, $S(\sigma) = \sigma$, $\varepsilon(\sigma) = 1$,

$(U^\sim.5)$ $\Delta(K_\lambda) = K_\lambda \otimes K_\lambda$, $S(K_\lambda) = K_\lambda^{-1}$, $\varepsilon(K_\lambda) = 1$,

$(U^\sim.6)$ $\Delta(E_i) = E_i \otimes 1 + K_{\alpha_i}\sigma^{p(i)} \otimes E_i$, $S(E_i) = -K_{\alpha_i}^{-1}\sigma^{p(i)}E_i$, $\varepsilon(E_i) = 0$.

Let $U^\sim_q b_+$ (resp. $U^\sim_q n_+$, $\mathbb{T}$) be an unital subalgebra generated by the elements $\{E_i\ (1 \le i \le n), K_\lambda\ (\lambda \in P_+)\}$ (resp. $\{E_i\ (1 \le i \le n)\}$, $\{K_\lambda\ (\lambda \in P_+)\}$).

Let $\mathbb{I}$ be the set of finite sequenses of $\{1, \ldots, n\}$. Put $E_{\mathrm{I}} = E_{i_1} E_{i_2} \cdots E_{i_p}$ for $\mathrm{I} = (i_1, i_2, \ldots, i_p) \in \mathbb{I}$ and put $E_\phi = 1$.

Lemma 2.1. As a **K** - vector space, $U^\sim_q b_+^\sigma$ has a basis elements such that

$E_{\mathrm{I}} K_\lambda \sigma^c$ $(\mathrm{I} \in \mathbb{I}, \lambda \in P_+, c \in \{0, 1\})$. In particular, we have

$$U^\sim_q b_+^\sigma \cong U^\sim_q n_+ \otimes \mathbb{T} \otimes \mathbf{K}\langle \sigma \rangle \text{ as } \mathbf{K}\text{-vector spaces.}$$

Proposition 2.2. There is a symmetric Hopf-pairing $<,>: U^{\sim}{}_q b_+{}^{\sigma} \otimes U^{\sim}{}_q b_+{}^{\sigma} \to K$ such that

(P.1) $\quad <\sigma , E_I K_\lambda \sigma^c> = \delta_{I\phi}\, (-1)^c ,$

(P.2) $\quad <K_\mu , E_I K_\lambda \sigma^c> = \delta_{I\phi}\, q^{(\mu,\lambda)} ,$

(P.3) $\quad <E_i , E_I K_\lambda \sigma^c> = \delta_{I\,(i)} .$

We put $I_{b+}{}^{\sigma} = \mathrm{Ker}<,>$ and put $u_q b_+{}^{\sigma} = U^{\sim}{}_q b_+{}^{\sigma} / I_{b+}{}^{\sigma}$.
Let $D\,(u_q b_+{}^{\sigma})$ be the quantum doble of $u_q b_+{}^{\sigma}$ with respect to $<,>$. For $X \in u_q b_+{}^{\sigma}$, we write X , X^{op} for $X \otimes 1 , 1 \otimes X \in D\,(u_q b_+{}^{\sigma})$ respectively.

Lemma 2.3. In $D\,(u_q b_+{}^{\sigma})$, the following equations hold:

$(D^{\sim}.1)$ $\sigma\cdot\sigma^{op} = \sigma^{op}\cdot\sigma ,\ \sigma K_\lambda{}^{op}\sigma = K_\lambda{}^{op} ,\ \sigma E_i{}^{op}\sigma = (-1)^{p(i)} E_i{}^{op} ,$

$\sigma^{op} K_\lambda \sigma^{op} = K_\lambda ,\ \sigma^{op} E_i \sigma^{op} = (-1)^{p(i)} E_i ,$

$(D^{\sim}.2)$ $K_\lambda \cdot K_\mu{}^{op} = K_\mu{}^{op} \cdot K_\lambda ,$

$K_\lambda E_i{}^{op} K_\lambda{}^{-1} = q^{-(\alpha_i,\lambda)} E_i{}^{op} ,\ K_\lambda{}^{op} E_i K_\lambda{}^{op-1} = q^{-(\alpha_i,\lambda)} E_i ,$

$(D^{\sim}.3)$ $E_i \cdot E_j{}^{op} - E_j{}^{op} \cdot E_i = \delta_{ij} (K_{\alpha_i}{}^{op} \sigma^{op\,p(i)} - K_{\alpha_i} \sigma^{p(i)}) .$

Let L be an ideal of K-algebra $D\,(u_q b_+{}^{\sigma})$ generated by $\sigma\cdot\sigma^{op} - \sigma^{op}\cdot\sigma$ and $K_\lambda \cdot K_\lambda{}^{op} - K_\lambda{}^{op} \cdot K_\lambda$ ($\lambda \in P_+$). It is clear that L is a Hopf-ideal. Put

$$u_q{}^{\sigma} = u_q{}^{\sigma}(\mathbb{E}.\Pi.p) = D\,(u_q b_+{}^{\sigma}) / L .$$

Put $u_q n_+ = U^{\sim}{}_q n_+ / (I_{b+}{}^{\sigma} \cap U^{\sim}{}_q n_+)$, $\mathfrak{t} = \mathbb{T} / (I_{b+}{}^{\sigma} \cap \mathbb{T})$.

Lemma 2.4. (1) As $\mathbf{K}$ -vector spaces,

$$u_q^{\sigma} \cong u_q n_+ \otimes t \otimes \mathbf{K}<\sigma> \otimes u_q n_+ \ (Xt\sigma^c Y^{op} \leftarrow X \otimes t \otimes \sigma^c \otimes Y).$$
$(c = 0, 1)$

(2) For $1 \le i \le N$, let $\gamma_i = \min\{\gamma \mid K_{\underline{\underline{\varepsilon}}_i}{}^{\gamma} = 1\} \in \mathbf{Z}_+ \cup \{+\infty\}$. Then the elements $K_{\underline{\underline{\varepsilon}}_1}{}^{\delta 1} \cdots K_{\underline{\underline{\varepsilon}}_N}{}^{\delta N}$ $(0 \le \delta_i < \gamma_i)$ form a $\mathbf{K}$ -basis of t .

(3) Let $\dot{u}_q$ be an unital subalgebra of u_q^{σ} generated by the elements $\{E_i, \dot{F}_i = E_i^{op}\sigma^{p(i)}\ (1 \le i \le n), K_\lambda\ (\lambda \in P_+)\}$. Then there is a Hopf-superalgebra structure on $\dot{u}_q$ with coproduct $\dot{\Delta}$ defined by

$$\dot{\Delta}(K_\lambda) = K_\lambda \otimes K_\lambda\ , \ \dot{\Delta}(E_i) = E_i \otimes 1 + K_{\alpha i} \otimes E_i\ , \ \dot{\Delta}(\dot{F}_i) = \dot{F}_i \otimes K_{\alpha i}{}^{-1} + 1 \otimes \dot{F}_i\ .$$

Theorem 2.5. Assume that q is an indeterminate and $\mathbf{K} = \mathbf{C}(q)$. Suppose that $(\alpha_i, \alpha_i) > 0$, $(\alpha_i, \alpha_j) \le 0$ and $2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i) \in \mathrm{Z}$. Let $\underline{\underline{G}}$ be the Kac-Moody Lie algebra defined for $(\ ,\ ) : \mathbb{E} \times \mathbb{E} \to \mathbf{K}$ and Π . Then $\dot{u}_q$ is isomorphic to the Drinfeld-Jimbo quantized enveloping algebra $U_q(\underline{\underline{G}})$.
(Jimbo type)

Theorem 2.6. Let $\underline{\underline{G}}$ be the simple $\mathbf{C}$-Lie algebra. Suppose that Π is the set of the simple roots of $\underline{\underline{G}}$. Assume that $\mathbf{K} = \mathbf{C}$. Let ζ be an primitive m-th root of unity such that $m >> 1$. Then $\dot{u}_\zeta$ is isomorphic to the Lusztig's quantum group at root of unity $u_\zeta(\underline{\underline{G}})$.

Theorem 2.5 can be immediately proved by Proposition 2.4.1 in [T] . Theorem 2.6 also seems to be well-known. For example, see [R].

§3. Root Systems of Simple Lie Superalgebras.

Let $\mathbb{G}$ be simple Lie superalgebras of types A_{N-1}, B_N, C_N, D_N, F_4, G_3 . Let $(\mathbb{E}, \Pi, p)$ be a triple related to a root system of $\mathbb{G}$ .From now on, we only

treat triples $(\mathbb{E}, \Pi, p)$ of following *Dynkin diagrams*.
In the following diagrams, the element under i-th dot denotes the i-th simple root $\alpha_i \in \Pi$. The i-th dot $\times$ stands for o (resp. $\otimes$) if $(\alpha_i,\alpha_i) \neq 0$ (resp. $= 0$). If i-th dot is o , $\otimes$ or $\bullet$, then we define $p(\alpha_i) = 0, 0, 1$ respectively. We also define a diagonal matrix $\mathbb{D} = (d_1,\dots,d_n)$ such that $\mathbb{A} = \mathbb{D}^{-1}((\alpha_i,\alpha_j))$ is a Cartan matrix of $\mathbb{G}$.

(A_{N-1})

1 2 N-1
×——×—······——× , $(\underline{\varepsilon}_i,\underline{\varepsilon}_i) = \pm 1$, $\mathbb{D} = \mathrm{diag}(1,\dots,1)$,
$\underline{\varepsilon}_1-\underline{\varepsilon}_2$ $\underline{\varepsilon}_2-\underline{\varepsilon}_3$ $\underline{\varepsilon}_{N-1}-\underline{\varepsilon}_N$

(B_N)

1 2 N-1 N 1 2 N-1 N
×——×—······——×⟹o , ×——×—······——×⟹• ,
$\underline{\varepsilon}_1-\underline{\varepsilon}_2$ $\underline{\varepsilon}_2-\underline{\varepsilon}_3$ $\underline{\varepsilon}_{N-1}-\underline{\varepsilon}_N$ $\underline{\varepsilon}_N$ $\underline{\varepsilon}_1-\underline{\varepsilon}_2$ $\underline{\varepsilon}_2-\underline{\varepsilon}_3$ $\underline{\varepsilon}_{N-1}-\underline{\varepsilon}_N$ $\underline{\varepsilon}_N$
$(\underline{\varepsilon}_i,\underline{\varepsilon}_i) = \pm 1$, $\mathbb{D} = \mathrm{diag}(1,\dots,1,1/2)$,

(C_N)

1 2 N-1 N
×——×—······——×⟸o , $(\underline{\varepsilon}_i,\underline{\varepsilon}_i) = \pm 1$, $\mathbb{D} = \mathrm{diag}(1,\dots,1,2)$,
$\underline{\varepsilon}_1-\underline{\varepsilon}_2$ $\underline{\varepsilon}_2-\underline{\varepsilon}_3$ $\underline{\varepsilon}_{N-1}-\underline{\varepsilon}_N$ $2\underline{\varepsilon}_N$

(D_N)

N-1 N-1
1 2 N-2 o $\underline{\varepsilon}_{N-1}-\underline{\varepsilon}_N$ 1 2 N-2 $\otimes$ $\underline{\varepsilon}_{N-1}-\underline{\varepsilon}_N$
×——×—···——×——o N , ×——×—···——×——$\otimes$ N ,
$\underline{\varepsilon}_1-\underline{\varepsilon}_2$ $\underline{\varepsilon}_2-\underline{\varepsilon}_3$ $\underline{\varepsilon}_{N-1}-\underline{\varepsilon}_N$ $\underline{\varepsilon}_{N-1}+\underline{\varepsilon}_N$ $\underline{\varepsilon}_1-\underline{\varepsilon}_2$ $\underline{\varepsilon}_2-\underline{\varepsilon}_3$ $\underline{\varepsilon}_{N-1}-\underline{\varepsilon}_N$ $\underline{\varepsilon}_{N-1}+\underline{\varepsilon}_N$
$(\underline{\varepsilon}_i,\underline{\varepsilon}_i) = \pm 1$, $\mathbb{D} = \mathrm{diag}(1,\dots,1,1)$,

(F_4)

1 4 3 2
o——o⟹o——$\otimes$,
$\underline{\varepsilon}_2-\underline{\varepsilon}_3$ $\underline{\varepsilon}_3-\underline{\varepsilon}_4$ $\underline{\varepsilon}_1$ $(\underline{\varepsilon}_1-\underline{\varepsilon}_2-\underline{\varepsilon}_3-\underline{\varepsilon}_4)/2$

$(\underline{\varepsilon}_1,\underline{\varepsilon}_1) = 6$, $(\underline{\varepsilon}_2,\underline{\varepsilon}_2) = (\underline{\varepsilon}_3,\underline{\varepsilon}_3) = (\underline{\varepsilon}_4,\underline{\varepsilon}_4) = -2$, $\mathbb{D} = \mathrm{diag}(2,1,1,2)$,

$$\begin{array}{lccc} & 1 & 3 & 2 \\ (\mathbf{G_3}) & \otimes\!\!-\!\!-\!\!-\!\!-\!\!\mathrm{o} & \!\!<\!\!\equiv\!\!\equiv\!\!\equiv & \!\!\mathrm{o}\ , \\ & \underline{\varepsilon}_1-\underline{\varepsilon}_2 & (\underline{\varepsilon}_2-\underline{\varepsilon}_3)/2 & \underline{\varepsilon}_3 \end{array}$$

$(\underline{\varepsilon}_1,\underline{\varepsilon}_1) = -2$, $(\underline{\varepsilon}_2,\underline{\varepsilon}_2) = 2$, $(\underline{\varepsilon}_3,\underline{\varepsilon}_3) = -6$, $\mathbb{D} = \mathrm{diag}(1,3,1)$.

§4. Defining relations of $u_q{}^{\sigma}(\mathbb{E}.\Pi.p)$ *of Simple Lie Superalgebras* $\mathbb{G}$.

Here we give defining relations of $u_q n_+$ of $u_q{}^{\sigma}(\mathbb{E}.\Pi.p)$ (see Lemma 2.4) when q is not a root of unity.

Put $P_+ = \mathbf{Z}\alpha_1 \oplus \cdots \oplus \mathbf{Z}\alpha_N$. We extend p to $p : P_+ \to \mathbf{Z}/2\mathbf{Z}$ additively. For $\delta = m_1\alpha_1 + \cdots + m_N\alpha_N \in P_+$, let $(u_q n_+)_\delta$ be a $\mathbf{K}$-subspace of $u_q n_+$ spaned by elements $E_{i_1}E_{i_2}\cdots E_{i_p}$ $(\#\{i_a = i\} = m_i)$. Then we have $u_q n_+ = \oplus_{\delta \in P_+} (u_q n_+)_\delta$. For $\delta, \nu \in P_+$ and $X_\delta \in (u_q n_+)_\delta$, $X_\nu \in (u_q n_+)_\nu$, put

$$\mathrm{ad}_{[\ ,\ ]}X_\delta(X_\nu) = [X_\delta\ , X_\nu] = X_\delta X_\nu - (-1)^{p(\delta)p(\nu)}q^{-(\delta,\nu)}X_\nu X_\delta .$$

Theorem 4.1. Let $(\mathbb{E}.\Pi.p)$ be a triple introduced in §3. Assume that q is not a root of unity. Let $u_q n_+$ be of be of $u_q{}^{\sigma}(\mathbb{E}.\Pi.p)$ (see Lemma 2.4) . Then, as $\mathbf{K}$-algebra, $u_q n_+$ is defined with the generators E_i $(1 \le i \le n)$ and the relations:

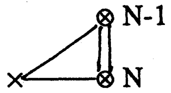
(r1) $[E_i\ , E_j] = 0$ if $(\alpha_i,\alpha_j) = 0$,

(r2) $(\mathrm{ad}_{[\ ,\ ]}E_i)^{m_{ij}}(E_j) = 0$ if $(\alpha_i,\alpha_i) \neq 0$ and $m_{ij} = 2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i) \in Z$,

(r3) $(\mathrm{ad}_{[\ ,\ ]}E_N)^3(E_{N-1}) = 0$ if $\overset{N-1}{\times}\!\!\Longrightarrow\!\!\overset{N}{\bullet}$,

(r4) $[\,[\,[E_i\ , E_j\,]\ , E_k\,], E_j\,] = 0$

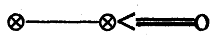
if $\overset{i}{\times}\!\!-\!\!-\!\!\overset{j}{\otimes}\!\!-\!\!-\!\!\overset{k}{\times}$, $\overset{i}{\times}\!\!-\!\!-\!\!\overset{j}{\otimes}\!\!\Longrightarrow\!\!\overset{k}{\mathrm{o}}$ or $\overset{i}{\times}\!\!-\!\!-\!\!\overset{j}{\otimes}\!\!\Longrightarrow\!\!\overset{k}{\bullet}$, (r5) $[[E_{N-2}, E_{N-1}], E_N] = [[E_{N-2}, E_N], E_{N-1}]$

if 

,

(r6) $[[[E_{N-2}, E_{N-1}], [[E_{N-2}, E_{N-1}], E_N]], E_{N-1}] = 0$

N-2 N-1 N

if  ,

(r7) $[[[[[[E_{N-3}, E_{N-2}], E_{N-1}], E_N], E_{N-1}], E_{N-2}], E_{N-1}] = 0$

N-3 N-2 N-1 N

if  .

§5. Root vectors of $u_q^{\sigma}(\mathbb{E}.\Pi.p)$ *of Simple Lie Superalgebras* $\mathbb{G}$.

Here we assume that there is $m >> 1$ satisfying $q^{\underline{\underline{m}}} \neq 1$ for $1 \leq \underline{\underline{m}} \leq m$. Assume that $(\mathbb{E}.\Pi.p)$ is the triple in §3. Let Φ be the set of roots of $\mathbb{G}$ and Φ_+ the set of positive roots with respect to Π. Let Φ_+^{red} be the set of positive roots defined by
$\Phi_+^{red} = \{\beta \in \Phi_+ \mid \beta/2 \notin \Phi_+\}$. For $\beta = c_1\alpha_1 + \dots + c_N\alpha_N \in P_+$, put
$ht(\beta) = c_1 + \dots + c_N$, $g(\beta) = \min\{i \mid i \neq 0\}$ and $c_\beta = c_{g(\beta)}$.

Define a half integer $\underline{\underline{ht}}(\beta)$ by $\underline{\underline{ht}}(\beta) = ht(\beta)/c_\beta$. For $\alpha, \beta \in P_+$, we say that $\alpha < \beta$ if they satisfy one of the following $\in \frac{1}{2}\mathbb{Z}$

(1) $g(\alpha) < g(\beta)$,

(2) $g(\alpha) = g(\beta)$ and $\underline{\underline{ht}}(\alpha) < \underline{\underline{ht}}(\beta)$,

(3) Π is of type D_N, $p(\underline{\underline{\varepsilon}}_i - \underline{\underline{\varepsilon}}_N) = 0$ and $\alpha = \underline{\underline{\varepsilon}}_i - \underline{\underline{\varepsilon}}_N$, $\beta = 2\underline{\underline{\varepsilon}}_i$ or

$\alpha = 2\underline{\underline{\varepsilon}}_i \ , \beta = \underline{\underline{\varepsilon}}_i + \underline{\underline{\varepsilon}}_N$ or $\alpha = \underline{\underline{\varepsilon}}_i - \underline{\underline{\varepsilon}}_N \ , \beta = \underline{\underline{\varepsilon}}_i + \underline{\underline{\varepsilon}}_N$.

We define q-root vectors E_β ($\beta \in \Phi_+^{red}$) of $u_q n_+$ of $u_q^{\sigma}(\mathbb{E}.\Pi.p)$ as follows.

Definition 5.1. For $\beta \in \Phi_+^{red}$, we define the element $E_\beta \in u_q n_+$ as follows. (For type F_4 , (resp. G_3), we write E_{abcd} and $\dot{E}_{abcd}$ (resp. E_{abc} and $\dot{E}_{abc}$) for $E a\alpha_1 + b\alpha_4 + c\alpha_3 + d\alpha_2$ and $\dot{E} a\alpha_1 + b\alpha_4 + c\alpha_3 + d\alpha_2$ (resp. $E a\alpha_1 + b\alpha_3 + c\alpha_2$ and $\dot{E} a\alpha_1 + b\alpha_3 + c\alpha_2$).

(1) We put $E\alpha_i = E_i \ (1 \le i \le n)$.
(2) Let $\alpha \in \Phi_+^{red}$ and $1 \le i \le n$ be such that $g(\beta) < i$ and $\alpha + \alpha_i \in \Phi$. Put $\dot{E}\alpha{+}\alpha_i = [\dot{E}_\alpha \ , E_i]$. If Π is of type B_N , $i = N$ and $\alpha = \underline{\underline{\varepsilon}}_j \ (1 \le j \le N\text{-}1)$, let $E\alpha{+}\alpha_N = (q^{1/2} + q^{-1/2})^{-1}\dot{E}\alpha{+}\alpha_N$. If Π is of type D_N , $i = N$ and $\alpha = \alpha_{N-1}$, let $E\alpha{+}\alpha_N = (q + q^{-1})^{-1}\dot{E}\alpha{+}\alpha_N$. If Π is of type F_4 , let $E_{1120} = (q + q^{-1})^{-1}\dot{E}_{1120}$ and $E_{1232} = (q^2 + 1 + q^{-2})^{-1}\dot{E}_{1232}$. If Π is of type G_3 , let $E_{121} = (q + q^{-1})^{-1}\dot{E}_{121}$, $E_{021} = (q + q^{-1})^{-1}\dot{E}_{021}$ and $E_{031} = (q^2 + 1 + q^{-2})^{-1}\dot{E}_{031}$. Otherwise, put $E\alpha{+}\alpha_i = \dot{E}\alpha{+}\alpha_i$.

(3) Let $\alpha ,\beta \in \Phi_+^{red}$ such that $g(\alpha) = g(\beta)$, $\alpha < \beta$, $\underline{\underline{ht}}(\beta) - \underline{\underline{ht}}(\alpha) \le 1$ and $\alpha + \beta \in \Phi_+^{red}$. Put $\dot{E}_{\alpha+\beta} = [\dot{E}_\alpha \ , \dot{E}_\beta]$. If Π is of type C_N (resp. D_N, F_4 or G_3), then $E_{\alpha+\beta}$ is defined by $(q + q^{-1})^{-1}\dot{E}_{\alpha+\beta}$ (resp. $(q + q^{-1})^{-1}\dot{E}_{\alpha+\beta}$, $(q^2 + q^{-2})^{-1}\dot{E}_{\alpha+\beta}$ or $(q^2 + 1 + q^{-2})^{-1}\dot{E}_{\alpha+\beta}$) .

By using similar computations in [Y2], we have

Proposition 5.2. (1) As a K-vector space, $u_q n_+$ is spaned by the elements

$$\prod^{<}_{\alpha \in \Phi_+^{red}} E_\alpha^{n_\alpha} \qquad (n_\alpha \in \mathbf{Z}_+ \text{ if } (\alpha,\alpha) \neq 0,\ n_\alpha = 0, 1 \text{ if } (\alpha,\alpha) \neq 0).$$

Here $\prod^{<}_{\alpha \in \Phi_+^{red}}$ denotes a product taken with a total order on Φ_+^{red} compatible with the partial order $<$.

(2)

$$< \prod^{<}_{\alpha \in \Phi_+^{red}} E_\alpha^{n_\alpha},\ \prod^{<}_{\alpha \in \Phi_+^{red}} E_\alpha^{m_\alpha} >$$

$$= \prod^{<}_{\alpha \in \Phi_+^{red}} \delta_{n_\alpha m_\alpha} \psi(n_\alpha\,;\,(-1)^{p(\alpha)} q^{(\alpha,\alpha)}) <E_\alpha, E_\alpha>^{n_\alpha} .$$

Here $\psi(n\,;\,t) = \Pi_{1 \le i \le n} \{(t^i - 1)/(t-1)\}$.

§6. *Poincaré-Birkhoff-Witt type Theorem* of $u_q^\sigma(\mathbb{E}.\Pi.p)$ *of Simple Lie Superalgebras* $\mathbb{G}$.

Define $d_\alpha \in (1/2)\mathbf{Z}_+$ by $d_\alpha = |(\alpha,\alpha)|/2$ if $(\alpha,\alpha) \neq 0$, $d_\alpha = 2$ if Π is of type G_3 and $\alpha = \alpha_1 + 2\alpha_3 + c\alpha_2$, $d_\alpha = 1$ otherwise.

For $\alpha = c_1\alpha_1 + \dots + c_N\alpha_N \in P_+$, put

$b(\alpha) = (q^{d\alpha} - q^{-d\alpha})<E_\alpha , E_\alpha>/\Pi_{1\leq i\leq n}(q^{di} - q^{-di})^{ci}$ and

$\gamma_\alpha = \min\{\gamma \mid \psi(\gamma ; (-1)^{p(\alpha)}q^{(\alpha,\alpha)}) = 0\} \in \mathbf{Z}_+ \cup \{+\infty\}.$

Lemma 6.1. $b(\alpha)$ can be written as $(-1)^a q^b$ for some $a , b \in \mathbf{Z}_+$. (For the precise value of $b(\alpha)$, see [Y2; Lemma 10 .3.1]).

By Proposition 5.2 and Lemma 6.1, we have:

Theorem 6.2. (*PBW-type theorem*) The elements

$$\overset{<}{\prod_{\alpha\in\Phi_+^{red}}} E_\alpha^{\delta_\alpha} \qquad (0 \leq \delta_\alpha < \gamma_\alpha)$$

form a K-basis of $u_q n_+$.

Proposition 6.3. Let $m > 10$ and ζ a primitive m-th root of unity. Then, as K-algebra, $u_\zeta n_+$ is defined with the generators E_i $(1\leq i\leq n)$ and the relations (r1-7) in Theorem 4.1 and relations

$$\text{(rr1)} \qquad E_\alpha^{\gamma_\alpha} = 0 \quad (\alpha \in \Phi_+^{red}) .$$

of

§7. Universal R-matrix $u_\zeta^\sigma(\mathbb{E}.\Pi.p)$ *of Simple Lie Superalgebra* $\mathbb{G}$.

Keep notation in §3-6. For $\alpha = c_1\alpha_1 + \cdots + c_N\alpha_N \in P_+$, put

$F_\alpha = (\Pi_{1\leq i\leq n}(q^{-di} - q^{di})^{ci})^{-1}(E_\alpha)^{op}\sigma^{p(\alpha)}$ (see Lemma 4.2) and

$u(\alpha) = (-1)^{ht(\alpha)}/b(\alpha)$.

Theorem 7.1. (*Universal R-matrix of* u_ζ^σ) Keep notation in Proposition 6.3.

$\in \; u_\zeta^\sigma \otimes u_\zeta^\sigma$

The Universal R-matrix $\boldsymbol{R}$ of $u_\zeta{}^\sigma = u_\zeta{}^\sigma(\mathbb{E}.\Pi.p)$ is given by

$$\boldsymbol{R} = \Big\{ \prod^{<}_{\alpha \in \Phi_+{}^{red}} \Big(\sum_{0 \le \delta_\alpha < \gamma_\alpha} \Big(\frac{\big((q^{d\alpha} - q^{-d\alpha})u(\alpha)E_\alpha \otimes F_\alpha \sigma^{p(\alpha)}\big)^{\delta_\alpha}}{\psi(n_\alpha \,;\, (-1)^{p(\alpha)} q^{(\alpha,\alpha)})} \Big)\Big)\Big\}$$

$$\cdot \Big\{1/2 \sum_{0 \le c,\, d \le 1} (-1)^{cd} \sigma^c \otimes \sigma^d \Big\} \cdot \prod_{1 \le i \le N} \Big\{ (1/\gamma_i) \sum_{0 \le \delta_i,\, \phi_i < \gamma_i} \zeta^{-(\underline{\underline{\varepsilon}}_i, \underline{\underline{\varepsilon}}_i)\delta_i \phi_i} K_{\underline{\underline{\varepsilon}}_i}{}^{\delta_i} \otimes K_{\underline{\underline{\varepsilon}}_i}{}^{\phi_i} \Big\}$$

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